Solutions 6: Quantum Channels (part 3))

Class problems

1. a) Prove that a linear map $\mathcal{E}$ is completely positive iff $(\mathcal{E} \otimes \mathbb{I})(|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|)$ is positive.
   b) Hence show that i. the dephasing channel is completely positive but ii. the transpose operation is not.

   Answer: a) Notice that a linear map $\mathcal{E}$ is complete positive iif $\mathcal{E} \otimes \mathbb{1}$ is positive.

   First, we show that $\mathcal{E}$ is completely positive $\implies$ $(\mathcal{E} \otimes \mathbb{I})(|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|)$ is positive. It is easy to show that $|vec(\mathbb{1})\rangle\langle vec(\mathbb{1})|$ is positive so obviously if $\mathcal{E}$ is completely positive, then $J(\mathcal{E})$ is positive (by definition $\mathcal{E}$ completely positive $\implies$ $\mathcal{E} \otimes \mathbb{1}$ positive $\iff$ $\mathcal{E} \otimes \mathbb{1}(O)$ is positive for any positive O).

   Proving the other direction ($\impliedby$) is slightly more complicate. If $J(\mathcal{E})$ is positive then it can be written as $J(\mathcal{E}) = \sum_k \lambda_k |e_k\rangle\langle e_k|$ where $\lambda_k \geqslant 0$ (where λ_k is the eigenvalue associated to eigenvector $|e_k\rangle$).

   We also have $J(\mathcal{E}) = \sum_{i,j} \mathcal{E}(|i\rangle\langle j|) \otimes |i\rangle\langle j|$, so we have $\mathcal{E}(|i\rangle\langle j|) = (\mathbb{1} \otimes \langle i|)J(\mathcal{E})(\mathbb{1} \otimes |j\rangle)$. Now we define the operators A_k by $(\mathbb{1} \otimes \langle i|)\sqrt{\lambda_k}|e_k\rangle = A_k|i\rangle$ which leads to $\mathcal{E}(|i\rangle\langle j|) = \sum_k A_k|i\rangle\langle j|A_k^\dagger$ and thus for any positive operator O we have $\mathcal{E} \otimes \mathbb{1}(O) = \sum_k (A_k \otimes \mathbb{1})O(A_k^\dagger \otimes \mathbb{1})$ which is positive because if $\langle\psi|O|\psi\rangle \geqslant 0$ for all $|\psi\rangle$ then its also the case (positive) for any non normalised vectors i.e. for any $|\tilde{\psi}_k\rangle = (A_k^\dagger \otimes \mathbb{1})|\psi\rangle$, so $\mathcal{E} \otimes \mathbb{1}$ is positive and thus $\mathcal{E}$ is completely positive (note that this is not the only way of proving it).

   b) i. The dephasing channel reduces the off-diagonal elements of a quantum state. Let us assume that the off-diagonal elements are multiplied by a factor $0 \leqslant \alpha < 1$ ($\alpha = 1$ is just the identity operation and $\alpha = 0$ is the completely dephasing channel). It is easy to show that $J(\mathcal{E})$ has eigenvalues α and 1 which are positive. Indeed, $J(\mathcal{E}) = \sum_{i\neq j} \alpha|ij\rangle\langle ij| + \sum_i |ii\rangle\langle ii|$. Hence, $J(\mathcal{E})$ is positive for the dephasing channel i.e. the dephasing channel is completely positive.

   ii. Here $J(\mathcal{E})$ is the $SWAP$ operator which is not positive indeed -1 is eigenvalue of the $SWAP$ operator associated to eigenvectors $|\psi^-_{i,j}\rangle = \frac{1}{\sqrt{2}}|i,j\rangle - |j,i\rangle)$ for all $i \neq j$. (Recall: $SWAP = \sum_{i,j} |ij\rangle\langle ji|$.)

2. (Optional - just if you want more practise) Consider the other quantum channels from last week's problem sheet. These were

$$\mathcal{E}(\rho) = p_0\rho + p_1 X\rho X + p_2 Y\rho Y + p_3 Z\rho Z \tag{1}$$

   the channel associated with the non-normalization Kraus operators

$$A_0 \propto \begin{pmatrix} 2 & 0 \\ 0 & 1 \\ 0 & 1 \\ 0 & 0 \end{pmatrix} \quad A_1 \propto \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 1 & 0 \\ 0 & 2 \end{pmatrix} \tag{2}$$

   and the channel induced on the system quantum A by the unitary

$$U = \frac{1}{\sqrt{2}}\left(X_A \otimes \mathbb{I}_B + Y_A \otimes X_B\right) . \tag{3}$$

   assuming the environment qubit starts in the state $|0\rangle$.

   a) Write out the Choi matrices for these three channels.

   b) Hence (or otherwise) find a (different) set of Kraus operators to represent the same channels.

   c) Consider the operation

$$\mathcal{E}(\rho) = 1/3(\alpha \operatorname{Tr}[\rho]\mathbb{I} + \beta \rho^T) .$$

   For what values of α and β does this operation output a normalized quantum state? For what values is it completely positive?

d) For the case where $\mathcal{E}$ represents a genuine quantum channel state a minimal Kraus representation for the channel.

e) Hence state a more general expression for any set of Kraus operators that can represent

Answer:
a) The Choi matrix for a channel with Kraus operators $\{A_i\}$ is given by $J(\mathcal{E}) = \sum_i |\text{vec}(A_i)\rangle\langle\text{vec}(A_i)|$. In the computational basis for the polarizing channel we have $J(\mathcal{E}) = \begin{pmatrix} p_0+p_3 & 0 & 0 & p_0-p_3 \\ 0 & p_1+p_2 & p_1-p_2 & 0 \\ 0 & p_1-p_2 & p_1+p_2 & 0 \\ p_0-p_3 & 0 & 0 & p_0+p_3 \end{pmatrix}$

For the Kraus operators A_0 and A_1 (with the normalization factor), we have $J(\mathcal{E}) = \frac{1}{6}\begin{pmatrix} 4&0&0&2&0&2&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&1&0&1&0&0&2 \\ 2&0&0&1&0&1&0&0 \\ 0&0&1&0&1&0&0&2 \\ 2&0&0&1&0&1&0&0 \\ 0&0&0&0&0&0&0&0 \\ 0&0&2&0&2&0&0&4 \end{pmatrix}$.

For the channel associated to the Stinespring dilation unitary in Eq.(3), we have $J(\mathcal{E}) = |01\rangle\langle 10| + |01\rangle\langle 10|$.

b) See part e).

c) This operation is trace preserving and positive iif

$$\begin{aligned} \frac{\alpha \text{Tr}[\mathbb{1}] + \beta}{3} &= 1 \quad \text{(trace preserving)} \\ 0 \leqslant \alpha \leqslant \frac{3}{\text{Tr}[\mathbb{1}] - 1} & \quad \text{(+positivity)} , \end{aligned} \tag{4}$$

where $\text{Tr}[\mathbb{1}] = d$. The trace preserving condition is straightforward by imposing $\text{Tr}[\rho] = \text{Tr}[\mathcal{E}(\rho)]$. For the positivity, we assume that ρ is positive (i.e. eigenvalues are $\geqslant 0$). Then, for any state $|\psi\rangle$ we have $\langle\psi|\mathcal{E}(\rho)|\psi\rangle = 1/3(\alpha\,\text{Tr}[\rho] + \beta\,\langle\psi|\rho^T|\psi\rangle) \geqslant 0$ iif $(\alpha + \beta\,\frac{\langle\psi|\rho^T|\psi\rangle}{\text{Tr}[\rho]}) \geqslant 0$, but by positivity of ρ we have $0 \leqslant \frac{\langle\psi|\rho^T|\psi\rangle}{\text{Tr}[\rho]} \leqslant 1$ for any $|\psi\rangle$ which leads to the result.

d) From previous answer we have the minimal set of Kraus operators given by

$$\left\{ \sqrt{\frac{\alpha+\beta}{3}}|i\rangle\langle i|, \sqrt{\frac{\alpha+\beta}{6}}\left(|i\rangle\langle j| + |j\rangle\langle i|\right), \sqrt{\frac{\alpha-\beta}{6}}\left(|i\rangle\langle j| - |j\rangle\langle i|\right) ; \forall(i,j) \text{ s.t. } i \in \{1,2,...,d\} \text{ and } j \in \{i+1,..,d\} \right\} .$$

e) From the lecture, we have that for any set of Kraus operators $\{A_k\}$ the set $\{B_i\}$ such that $B_i = \sum_k u_{i,k} A_k$ where $u_{i,k}$ are matrix element of a unitary/isometry is also a set of Kraus that describe the same channel. (Notice that as it's valid for isometries, then the size of set $\{B_i\}$ might be larger than $\{A_k\}$ as expected.)